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INTRODUCTION.

It has seemed to the Editors that there is not only room but a real need for a mathematical Journal of the character and scope of this MONTHLY. At the present time there is no Mathematical Journal published in the United States sufficiently elementary to appeal to any but a very limited constituency, and that comes to its readers at regular intervals. Most of our existing Journals deal almost exclusively with subjects beyond the reach of the average student or teacher of Mathematics or at least with subjects with which they are not familiar, and little, if any space, is devoted to the solution of problems. While not neglecting the higher fields of mathematical investigation, THE AMERICAN MATHEMATICAL MONTHLY will also endeavor to reach the average mathematician by devoting regular departments to the important branches of Mathematical Science.

It is recognized that those improvements in the Science are most fruitful, which lead to improvements in the elementary treatises, and yet it must be admitted that little has been accomplished by previous mathematical journals in this line, as the crudities and solecisms handed down from one text-book to another bear witness.

While realizing that the solution of problems is one of the lowest forms of Mathematical research, and that, in general, it has no scientific value, yet its educational value can not be over estimated. It is the ladder by which the mind ascends into the higher fields of original research and investigation. Many dormant minds have been aroused into activity through the mastery of a single problem.

THE AMERICAN MATHEMATICAL MONTHLY will, therefore, devote a due portion of its space to the solution of problems, whether they be the easy problems in Arithmetic, or the difficult problems in the Calculus, Mechanics, Probability, or Modern Higher Mathematics. Papers and other interesting features will be presented, including portraits of prominent mathematicians and their biographies, a column of Queries and Information in which readers may have informa-

tion furnished and their doubts cleared up by the aid of the contributors and editors, a column of Notes, and Book reviews.

No pains will be spared on the part of the Editors to make this the most interesting and most popular journal published in America. In order to do this, we must have the earnest co-operation of our readers. Teachers, students and all lovers of mathematics are, therefore, cordially invited to contribute problems, solutions and papers on interesting and important subjects in mathematics. We will be pleased to note your success, and all information of interest in regard to our contributors will be cheerfully received and noted.

All problems, solutions, and articles intended for publication in the February Number, should be received on or before February 1st, 1894. Solutions to problems in this Number will appear in March Number, but should be mailed to Editors before February 15th.

It will be our aim to have the MONTHLY reach its subscribers about the middle of each month. If you do not receive your Number about that time, inform us immediately.

B. F. FINKEL,
J. M. COLAW, } EDITORS.

BIOGRAPHY.

PROF. E. B. SEITZ, M. L. M. S.

BY B. F. FINKEL.

Professor Enoch Beery Seitz, the most distinguished mathematician of his day, was born in Fairfield county, Ohio, August 24, 1846, and died at Kirksville, Missouri, October 8, 1883. His father, Daniel Seitz, was born in Rockingham county, Virginia, December 17, 1791, and was twice married. His first wife's maiden name was Elizabeth Hite, of Fairfield county, Ohio, by whom he had eleven children. His second wife's maiden name was Catherine Beery, born in the same county, April 11, 1808, whom he married April 15, 1832, and by whom he was blessed with four sons and three daughters. He died near Lancaster, Ohio, October 14, 1864, in his seventy-third year, having been a resident of Fairfield county for sixty-three years.

Prof. Seitz, the third son by his father's second marriage, passed his boyhood on a farm, and like most men that have become noted, had only the advantages of a common school education. Possessing, however, a great thirst for learning, he applied himself to his books in private, and became a very fine scholar in the English branches, especially excelling in arithmetic. In the common school, though yet a little boy, he had greater power of arithmetical analysis than some of his teachers. He completed algebra at the age of fifteen without an instructor. He chose teaching as his profession, which he followed for a number of years with gratifying success.

He took a mathematical course in the Ohio Wesleyan University in 1870, but did not finish it or graduate. In 1872, he was elected one of the teachers in the Greenville High School, which position he held till 1879. On the 24th of June, 1875, he married Miss Anna E. Kerlin, one of Dark county's most refined ladies. In 1879, he was elected to the chair of mathematics in the Missouri State Normal school, Kirksville, Missouri, which position he held till death called him from the confines of earth, ere his star of fame had reached the zenith of its glory. He was striken by that "demon of death," typhoid fever, and passed the mysterious shades, to be numbered with the silent majority, on the 8th of October, 1883. On March the 11th, 1880, he was elected a member of the London Mathematical Society, being the fifth American so honored.

Prof. Seitz was in mathematics what Demosthenes was in oratory; Shakespeare in poetry; and Napoleon in war: the equal of the best, the peer of all the rest.

He began his mathematical course in 1872 by contributing solutions to the problems proposed in the "Stairway" department of the *Schoolday Magazine*, conducted by Artemas Martin. His masterly and original solutions to difficult Average and Probability problems, soon attracted universal attention among mathematicians.

Dr. Martin, being desirous to know what works he had treating on that difficult subject, was greatly surprised to learn that he had no works upon the subject, but had learned what he knew about that difficult department of mathematical science by studying the problems and solutions in the *Schoolday Magazine*. He then contributed to the *Analyst*, the *Mathematical Visitor*, the *Mathematical Magazine*, the *School Visitor*, and the *Educational Times*, of London, England.

In each of these journals, Prof. Seitz was second to none, as his logical and

classic solutions to Average and Probability problems, rising as so many monuments to his untiring patience and indomitable energy and perseverance will attest.

"His name first appears as a contributor to the *Educational Times* in Vol. XVII., of the *Reprint*, year 1873. To Vol. XXI., he contributed a solution of the problem, "Find the average area of a spherical triangle." His solution takes up just twelve lines, and it was the only solution received by the editor.

In Vol. XXXII., p. 105, he solved the problem, "If A, B, C, and D are four points taken at random in the surface of a given circle; show that the chance that E, the intersection of the straight lines through A, B and C, D, lies between A, B and between C, D, is $\frac{1}{3} - \frac{35}{36\pi^2}$ ".

In this communication his genius is displayed to a grand advantage; he is at home in his favorite field of investigation. The answer requires the evaluation of an octuple integral. The work is ably done and furnishes a fine specimen of what a classic solution ought to be. To Vol. XXXIX., he contributed a number of problems and solutions, and three solutions of problems proposed by other contributors; three have his own solution appended—no others, apparently, having been received by the editor. They look at first sight like a forest of definite integral symbols, but they are evidently in the line of his favorite pursuit, "Average and Probability." The *Mathematical Visitor* is adorned with some of his choicest solutions, which ever display his mathematical genius. His first solution for the *Mathematical Visitor* is of the following problem: "Find the equation to the locus of the centers of all the circles that can be incribed in a given semi-ellipse." The solution undoubtedly required a vast amount of patience and profound insight into the intricate and subtle relations which had to be traced in order to reach a result. The copying of the answer would exhaust the patience of the average student. On page 33, is a solution of the following problem: "A straight tree growing vertically on the side of a mountain was broken off by the wind, but not severed; find the chance that the top reaches the ground."

On page 37, is a solution to a prize problem. "A boy stepped upon a horizontal turn-table while it was in motion, and walked across it keeping all the time in the same vertical plane. The boy's velocity is supposed to be uniform in his track on the table, and the motion of the table toward him. The velocity of a point in the circumference of the turn-table is n times the volocity of the boy along the curve he describes. Required the nature of the curve the boy describes on the table, and the distance he walks while crossing it (1) when n is less than 1, (2) when n equals 1, (3) when n is greater than 1." His solution is a pantheon in grandeur and sublimity, adorned with the richest ornament of thought. Though the calculus acting as a radius vector, sweeping from one limit to another and embracing every element between the limits of that to which it is applied, yet through what a labyrinth of complex and hidden principles is the mind obliged to pass in order to see that the adjustment of this instrument will contain every element in its flight from one limit to another.

This solution alone would have been sufficient to place his name high in the category of American mathematicians. But this masterly solution is only one of the magnificent edifices of thought he erected for the children of men, in which they may congregate and learn something of the vastness and everlasting grandeur of its construction. On page 58 of the *Mathematical Visitor*, his name is attached to the solution of another "prize problem":

If $y=x+ax^2+bx^3+cx^4+$ &c. .....(1), and,

$x=y+Ay^2+By^3+Cy^4+$&c.....(2): find, A, B, C, D, E, F, G, H, I, J, K, L, M, N, P, in terms of a, b, c, d, e, f, g, h, i, j, k, l, m, n, p.

The principle involved in this problem is not the difficulty in effecting a solution: it is the prodigious amount of labor required in order to obtain a correct result. Any one acquainted with the process of reverting a series, knows that the work is tedious. But this wonderful mathematician, for whom no problem was too abstruse or labor too great, accomplished the work. His value of P covers about half of a quarto page. He finds the values of A, B, C, ... I, by actually performing the work, when the acuteness of his intellectual vision discerned a law by which the succeeding coefficients could be easily written out.

On page 79, is a solution to the problem: "Find the average distance between two points taken at random within a rectangular solid, edge, a, b, c." The solution, with a beautiful figure, covers an entire page, and is grand and imposing. On page 157, appears a fine solution to the problem: "If three dice be piled up at random on a horizontal plane, what is the probability that the pile will not fall down?"

On page 21, Vol. II, he has given a solution to the problem: "A cube is thrown into the air and a random shot is fired through it; find the chance that the shot passes through the opposite faces."

This problem had been proposed in 1864, by the great English mathematician, Prof. Woolhouse, who solved it with great labor. It was said by an eminent mathematician of that time, that the task of writing out a copy of that solution was worth more than the book in which it was published.

No other mathematician seemed to have the courage to investigate this problem after Prof. Woolhouse gave his solution to the world, till Prof. Seitz took it up and demonstrated it so elegantly in half a page of ordinary type, that he fairly astonished both the mathematicians of Europe and America.

Prof. Woolhouse was the best English authority on probabilities, even before Prof. Seitz was born.

It was the solution of this problem that won for Prof. Seitz the acknowledgment of his superior ability, in this abstruse department of mathematics, over any other man in either hemisphere. These are only a few of the many problems to which he has furnished the finest solutions. In studying his work, one is struck with the simplicity to which he has reduced the solutions of some of the most intricate problems. When he grasped a problem in its entirety, he had mastered all problems of that class. He would so vary the conditions in thinking of one special problem and in effecting a solution that he had generalized all similar cases, so exhaustive was his analyses. Behind the words he saw all the ideas represented. These he translated into symbols, and then he handled the symbols with a facility that has never been surpassed.

What he might have accomplished in his maturer years, no man may say; but at the age of thirty-seven he laid down his pen, and gave to God, from whence it came, the casement and the key of his mighty intellect, leaving his impress indelibly stamped upon the thinking and scientific world for all time. He has written his name in characters of gold and prismatic hues on the pinnacle of the temple of fame, and his good work will ever be cherished in the memories of those whom he has left behind.

He was a man of the most singularly blameless life; his disposition was amiable; his manner gentle and unobtrusive; and his decision, when circumstances demanded it, was prompt and firm as the rocks.

He did nothing from impulse; he carefully considered his course; and with a wise judgment came to conclusions that his conscience approved, and when his de-

cision was made, it was unalterable. He never made an open profession of religion, yet he was an intensely religious man. He rested his hopes on the sacrifice of the tender and loving Saviour, and we feel satisfied that he has entered into that rest which remaineth for the people of God.

Professor Seitz was not only a mathematician, but he was eminently proficient in other branches of knowledge. His mind was cast in a gigantic mold. "Being devout in heart as well as great in intellect, 'signs and quantities were to him but symbols of God's eternal truth' and he 'looked up through nature up to natures God.' Professor Seitz, in the very appropriate words of Dr. Peabody, regarding Benjamin Peirce, Professor of Mathematics and Astronomy in Harvard University, 'saw things precisely as they are seen by the infinite mind. He held the scales and compasses with which the eternal wisdom built the earth and meted out the heavens. As a mathematician, he was adored with awe. As a man, he was a christian in the whole aim and tenor of life.' " No mathematician was so universally loved and honored by his contemporaries as was Professor Seitz.

He did not gain his knowledge from books, for his library consisted of only a few books and periodicals. He gained such a profound insight into the subtle relation of numbers by close application with which he was particularly gifted. He was not a mathematical genius, that is, as usually understood, one who is born with mathematical powers fully developed; but he was a genius in that he was especially gifted with the power to concentrate his mind upon any subject he wished to investigate. This happy faculty of concentrating all his powers of mind upon one topic to the exclusion of all others and viewing that topic from all sides, enabled him to proceed with certainty. Thread by thread and step by step, he took up and followed out long lines of thought and arrived at correct conclusions. The darker and more subtle the question appeared to the average mind, the more eagerly he investigated it. No conditions were so complicated as to discourage him.

He left a wife and four sons, one of whom has gone to join his father in ths realms of eternal peace. His mother, now (1894) eighty-six years old is still living and enjoying good health.

LOWEST INTEGERS REPRESENTING SIDES OF A RIGHT TRIANGLE.

By LEONARD E. DICKSON, B. Sc., Fellow in Pure Mathematics, University of Texas.

Let the whole numbers expressing the lengths of the sides of a right-angled triangle be reduced to their lowest forms by dividing out their highest common divisor.

Call the resulting numbers a, b, and c.

1. They can not all be *even* numbers. For if so, they would still have the common divisor 2.

2. They can not all be *odd* numbers. For $a^2+b^2=c^2$; and, if a and b are odd, there squares are odd, and the sum of their squares even. But c^2 being even, c must be even.

3. c, the greatest of the three, must be *odd*; and, of the remaining two, one must be even and the other odd.

$b^2=c^2-a^2=(c+a)(c-a)$. Suppose b is even. The product of $(c+a)$ and $(c-a)$ is even; hence, both factors are even. [For the case in which one factor is even and the other odd is impossible, since the sum and the difference of two numbers are both even or both odd.] Hence, since the sum and difference of a and c are both even, they are themselves both even or both odd. But by 1, a and c can not both be even if b is even. Hence, when b is even, a and c are both odd.

Similarly, $a^2=c^2-b^2=(c-b)(c+b)$. Reasoning exactly as before, we find that when a is even, b and c are both odd. Hence, we may conclude that c is always odd; and that b is even when a is odd, but odd when a is even.

4. As a corollary to 3, we get the theorem: When the sides of a right triangle are expressed by integers, whether reduced by dividing out their common divisor or not so reduced, the perimeter of the triangle is expressed by an *even* number.

Formulae for obtaining values for a, b, and c.

5. *Rule of Pythagoras:* n, $\frac{n^2-1}{2}$, and $\frac{n^2+1}{2}$, when n is *odd*. The number $\frac{n^2-1}{2}$ is always divisible by 4. For, writing $(2k+1)$ for the odd number n, we get $\frac{n^2-1}{2}=\frac{4k^2+4k}{2}=2k(k+1)$, where $k(k+1)$ is even whether k be even or odd. As a corollary, the area of the triangle formed is expressed by an *even* number.

6. *Plato's Rule:* m, $\frac{m^2}{4}-1$, $\frac{m^2}{4}+1$, where m is *even*. THEOREM: m is always divisible by 4. For $\frac{m^2}{4}+1$ must [by 3.] be odd. Thus $\frac{m^2}{4}$ is even. Hence, m^2, being divisible by 8, must be divisible by 16, in order to be a perfect square. As before, the area of the triangle formed is expressed by an *even* number.

7. *Euclid's Rule:* $\sqrt{xy}$, $\frac{x-y}{2}$, $\frac{x+y}{2}$, where x and y are both even or both odd, and where xy is a perfect square. Further, x and y must have no common factor greater than 2; for if so, the sides a, b, and c would contain this factor.

8. *Rule of Maseres:* m, $\frac{m^2-n^2}{2n}$, $\frac{m^2+n^2}{2n}$. Since the last number is an integer, (m^2+n^2) is even and, hence, m and n are both even or both odd. m being one number, n is evidently the difference of the other two. [The numbers found by Maseres' Rule are usually expressed as (m^2-n^2), $2mn$, and (m^2+n^2), derived from those above by multiplying through by $2n$. As they thus contain a common factor, they are replaced by the ones given above, which are in their lowest form.]

Correlation of the Four Rules.

9. The Rule of Pythagoras and Plato's Rule are only special cases of the Rule of Maseres; while Euclid's Rule is that of Maseres under a different form. For let $n=1$ in the numbers found by the Rule of Maseres, viz.: m, $\frac{m^2-n^2}{2n}$, and $\frac{m^2+n^2}{2n}$, where m and n are both even or both odd, and we get the numbers given by the Rule of Pythagoras, viz.; m, $\frac{m^2-1}{2}$, $\frac{m^2+1}{2}$, where m is *odd*. Let $n=2$, and we get the numbers given by Plato's Rule; m, $\frac{m^2}{4}-1$, and $\frac{m^2}{4}+1$, where m is *even*. Let $n=y$ and $\frac{m^2}{n}=x$, and we get the numbers given by Euclid's Rule; $\sqrt{xy}$, $\frac{x-y}{2}$, $\frac{x+y}{2}$, where x and y are both even or both odd, and where xy is a perfect square (m^2).

A New Rule.

10. Let a, b, and c be the numbers reduced to their lowest form which represent the sides of a right-angled triangle. Let $(c-a)=m$; $(c-b)=n$, where m and n are integers, one being even and the other odd.

$$\therefore (c-m)^2+(c-n)^2=c^2,$$
$$c^2-2c(m+n)+m^2+n^2=o,$$
$$c^2-2c(m+n)+(m+n)^2=2mn,$$
$$c-(m+n)=\pm\sqrt{2mn},$$
$$c=m+n\pm\sqrt{2mn}.$$

Since $2c=(a+b)+(m+n)$ and $c<(a+b)$; $\therefore c>(m+n)$. Hence, we must take the plus sign before the radical.

$$\therefore c=m+n+\sqrt{2mn};$$
$$a=m+\sqrt{2mn};\text{ and}$$
$$b=n+\sqrt{2mn}.$$

By inspection, m and n must have no common factor; for this factor would appear in a, b, and c. My rule may be stated formally, thus:

Take any two integers, m and n, such that one is even and the other odd, and such that their product is a perfect square; then the three sides will be $m+\sqrt{2mn}$, $n+\sqrt{2mn}$, $m+n+\sqrt{2mn}$.

By Euclid's Rule (to which the other three are special cases or equivalent), we take any two integers, m and n, such that both are even or both odd, and such that their product is a perfect square; then the sides are $\sqrt{mn}$, $\frac{m-n}{2}$, $\frac{m+n}{2}$.

Complete Table of Values.

11. We may get an absolutely complete table of values for a, b, and c, and one in which each set is reduced to its lowest form, by employing either Euclid's Rule or my rule, in each case observing the necessary conditions and restrictions.

In Euclid's set of values, $\sqrt{xy}$, $\frac{x-y}{2}$, and $\frac{x+y}{2}$, x and y must both be even or both odd; and further they can have no common divisor higher than 2, since a, b, and c would then have this divisor. The *practical* rule, which gives every possible set of values in their lowest form, and which gives each set but once is as follows: Let y be any *odd* square number whatever, and x any greater *odd* square number not divisible by y. Thus if $y=9$, x may be 25, 49, 121, 169, etc.

In my set of values, $m+\sqrt{2mn}$, $n+\sqrt{2mn}$, $m+n+\sqrt{2mn}$, m and n must one be even and the other odd; and further, they can have no common divisor whatever, since a, b, and c would then have this divisor. My *practical* rule, which gives every possible set of values in their lowest form, and which gives this set but once, is this: Let m be any *odd* square number whatever, and let n be the *double* of *any* square number whatever not divisible by m. Thus if $m=9$, n may be the *double* of 1, 4, 16, 25, 49, 64, 100, etc.

Additional Theorems.

12. One out of each set of values found by my practical rule is divisible by 4. Let $m=(2l+1)^2$ and $n=2k^2$. Then $\sqrt{2mn}=2k(2l+1)$. Thus $n+\sqrt{2mn}=2k^2+2k(2l+1)=2k(2l+2)=4k(l+1)$.

13. One out of each set of numbers found by the practical rule derived from Euclid's Rule is divisible by 4. Let $y=(2l+1)^2$ and $x=(2k+2)^2$. $\frac{x-y}{2}=2(l^2-k^2+l-k)=2(l-k)(l+k+1)$. Now since $(l-k)$ and $(l+k)$ are both even or both odd, $(l-k)$ and $(l+k+1)$ are one even and the other odd. $\therefore \frac{x-y}{2}$ is divisible by 4.

14. As a corollary, the area of *any* right-angled triangle whose sides are integers is express by an *even* integer.

15. The radii of the inscribed and three escribed circles of any right-triangle whose sides are integers are all expressed by integers. The trigonometric expressions for r, r_a, r_b, r_c, viz.; $\frac{\triangle}{s}, \frac{\triangle}{s-a}, \frac{\triangle}{s-b}, \frac{\triangle}{s-c}$, become for the right-triangle $(s-c)$, $(s-b)$, $(s-a)$, and s respectively, where s is the semi-perimeter (always an integer). Hence, the radii are all integers.

Dr. Halsted's *Mensuration* contains a three-page table of values for a, b, and c. It is absolutely *complete* as far as the 59th set; but beyond this, sets are occasionally omitted.

SCALENE TRIANGLES.

Let a, b, and c be three integers having no common divisor which represent the sides of a scalene triangle whose *area* may be expressed by an *integer*.

1. They can not all be even.
2. They can not all be odd.

For $\triangle=\sqrt{s(s-a)(s-b)(s-c)}=\frac{1}{4}\sqrt{2(a^2b^2+a^2c^2+b^2c^2)-(a^4+b^4+c^4)}$. Now

if a, b, and c all be odd, $(a^4+b^4+c^4)$ is odd and the expression under the radical sign must be odd. Thus the area could not be expressed by an integer.

3. No two of them can be even. Proved as in (2).

4. Hence, of the three, one is even and the other two odd.

5. As a corollary, the perimeter of the triangle will be expressed by an *even* number.

6. THEOREM: The area of the triangle is expressed by an *even* number. Let $a=2l+1$; $b=2m+1$; $c=2n$, as one of the sides is even and the other two odd. $\therefore \frac{a+b+c}{2}=l+m+n+1$; $\frac{a+b-c}{2}=l+m-n+1$; $\frac{a-b+c}{2}=l-m+n$; $\frac{-a+b+c}{2}=-l+m+n$. Hence, of these four quantities, two are even and two odd. $\therefore$ area, $=\sqrt{\frac{a+b+c}{2}\cdot\frac{a+b-c}{2}\cdot\frac{a-b+c}{2}\cdot\frac{-a+b+c}{2}}$, is divisible by 2.

7. Problem: To find three integers a, b, and c representing the sides of a triangle whose *area* may be expressed by an *integer*.

$$\triangle=\tfrac{1}{4}\sqrt{2(a^2b^2+a^2c^2+b^2c^2)-(a^4+b^4+c^4)}.$$

Solving for a^2, we get $a^2=b^2+c^2\pm2\sqrt{b^2c^2-4\triangle^2}$.

Similarly, $b^2=a^2+c^2\pm2\sqrt{a^2c^2-4\triangle^2}$.

Similarly, $c^2=a^2+b^2\pm2\sqrt{a^2b^2-4\triangle^2}$.

Let a be greater than either b or c. Then $\angle A$ may be acute or obtuse; while $\angle B$ and $\angle C$ are both acute. Hence, writing l for $-\sqrt{b^2c^2-4\triangle^2}$, m for $-\sqrt{a^2c^2-4\triangle^2}$, n for $-\sqrt{a^2b^2-4\triangle^2}$, l may be $\pm$, while m and n *must* be $+$. Further, the conditions of the problem require that l, m, and n be integers. We have $a^2-b^2-c^2=\mp2l$; $b^2-a^2-c^2=-2m$; $c^2-a^2-b^2=-2m$. $\therefore$ $a^2=m+n$; $b^2=n\pm l$; $c^2=m\pm l$. Thus the conditions that $b^2c^2-4\triangle^2$, $a^2c^2-4\triangle^2$, $a^2b^2-4\triangle^2$, be perfect squares, l^2, m^2, and n^2, respectively, reduce to the single condition that $4\triangle^2=mn\pm ml\pm nl$. Hence, the Rule to find a, b, c such that area $\triangle ABC$ be integral is: Take any three integers $\pm l$, m, and n, such that one is odd and the other two are even, and such that the sums of them two at a time are perfect squares, and lastly such that the sum of their products two at a time is a perfect square; then $a=\sqrt{m+n}$; $b=\sqrt{m\pm l}$; $c=\sqrt{m\pm l}$; and $\triangle=\frac{1}{2}\sqrt{mn\pm ml\pm nl}$.

8. Let fall a $\perp$ from each vertex of the triangle upon the opposite side. We can express the intercepts on the sides either in terms of l, m, and n or of a, b, and c. For example, those on c are $\frac{\pm l}{\sqrt{m\pm l}}$ and $\frac{m}{\sqrt{m\pm l}}$ or in terms of a, b, c, $\frac{b^2+c^2-a^2}{2c}$ and $\frac{a^2+c^2-b^2}{2c}$. Thus in general these intercepts are not integers.

9. If two right-triangles, having a perpendicular side of the one equal to a perpendicular side of the other, be placed so that these sides coincide, the other perpendicular sides falling in the same straight, then one of the two possible scalene triangles are formed. The sides and the area of these scalene tri-

angles are expressed by integers, providing the three sides of each triangle be integers. But from (8) we find that the sides of the new triangles may have a common factor. Guarding this point, we may form from a table of the lowest integers representing sides of a right triangle a corresponding table for a scalene triangle whose *area is an integer*. A four-page table of this kind is given in Dr. Halsted's *Mensuration*.

POSTULATE I. OF EUCLID'S ELEMENTS.

By Professor JOHN L. LYLE, Ph, D., Westminister College, Fulton, Missouri.

"Let it be granted that a straight line may be drawn from any one point to any other point." Euclid lays down the statement just quoted as his first postulate regulative of geometrical constructions. Wherever any two points may be located in unbounded space, Euclid assumes that a straight line may be drawn from one of them to the other.

Such a line is finite in length, of course, according to the definition that "a *finite* straight line is one that has *two ends*".

The assumption that a straight line of *infinite* (boundless) length can be drawn between two points in space is not only anti-Euclidean but also destructive of the logical law of non-contradiction which forbids that contradictory marks shall be attributed to any straight line whatever. A line can not have two ends and at the same time be without ends, (infinite, that is, unbounded).

In every rectilineal triangle, there are three angular points and each of the three sides may be constructed in strict harmony with the 1st postulate of Euclid. Each of these sides has the distinctive marks of a finite straight line, to wit: two ends.

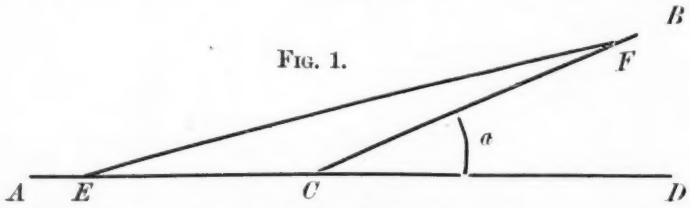
Euclid proves in proposition XVII, Book I., that "Any two angles of a triangle are together less than two right angles". This proposition is the converse of Euclid's 12th axiom about which so much has been written.

John Playfair states that axiom as follows: "If a straight line meet two straight lines, so as to make the interior angles on the same side of it less than two right angles, these straight lines being continually produced will at length meet on the side on which the angles are less than two right angles."

Lobatschewsky, in his theorem 19, demonstrates the proposition that the angle-sum of a rectilineal triangle can not be greater than two right angles. Then assuming the falsity of Euclid's 12th axiom, he concludes (whether logically or illogically) that the angle-sum of a rectilineal triangle is less than two right angles.

But a single angle ACB can always be drawn less than two right angles and in such manner as that it shall differ from two right angles by as small an angle a as we please. The single angle ACB may, therefore, be made equal to any assumed sum whatever (less than two right angles) of the three angles of a recti-

FIG. 1.



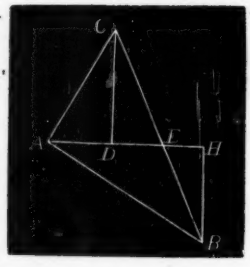
lineal triangle. Let the angle ACB, Fig. 1, be equal to the assumed angle-sum of a rectilineal triangle, to wit: two right angles—a. Then, a straight line can not be drawn from any point E on the leg AC to any point F on the leg CB; for if that is granted, we shall have a triangle ECF whose angle-sum is greater than two right angles—a, which is against the hypothesis. But to refuse to grant that a straight line may be drawn from any point E to any other point F is to discredit postulate 1, of Euclid's Elements. The assumptions of Lobatschewsky's geometry are at war with Euclid's 1st postulate.

DEMONSTRATION OF THE "TANGENT PROPOSITION," AND OTHERS.

By Professor P. H. PHILBRICK, M. S., C. E., Lake Charles, Louisiana.

Let ABC be the triangle considered. Represent the parts as usual. Make $CE = CA$ and draw CD and BH perpendicular to AE and AE prolonged.

We have $A = CAE + BAE$; and $B = AEC - BAE = CAE - BAE$. Adding and dividing by two, gives $CAE = \frac{1}{2}(A+B)$. Subtracting and dividing by two gives $BAE = \frac{1}{2}(A-B)$.



The triangles ACD, ECD, and BEH are similar.

Hence, $\frac{AD}{EH} = \frac{AC}{BE}$, $\frac{DE}{EH} = \frac{CE}{BE}$, and $\frac{EH}{EH} = \frac{BE}{BE}$.

Adding, we have $\frac{AD+DE+EH}{EH} = \frac{AC+CE+EB}{BE}$.

That is, $\frac{AH}{EH} = \frac{a+b}{a-b}$ or $\frac{EH}{AH} = \frac{a-b}{a+b}$.

Also, $\frac{BH}{AH} = \tan \frac{1}{2}(A-B)$ and $\frac{BH}{EH} = \tan \frac{1}{2}(A+B)$.

Dividing gives $\frac{EH}{AH}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$.

Equating, we have $\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$.....(1).

From triangle BAE, $\frac{BE}{AB}=\frac{\sin BAE}{\sin AEC}$, or $\frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}$...(2)

Also $AH=AD+DE+EH=AC\cos\frac{1}{2}(A+B)+CE\cos\frac{1}{2}(A+B)+EB\cos\frac{1}{2}(A+B)$, $=(a+b)\cos\frac{1}{2}(A+B)$.

In triangle BAH, $AH=c\cos\frac{1}{2}(A-B)$. Equating these values of AH gives $\frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}$....(3) The quotient of (2) divided by (3) gives (1) also.

CERTAIN SERIES OF INTEGRAL, RATIONAL, SCALENE TRIANGLES.

By SYLVESTER ROBINS, Long Branch Depot, New Jersey.

Investigation leads to the belief that there is an endless number of infinite series of integral, rational, scalene triangles, wherein the base of every term in nth series is $n+2$, and the difference between the other two sides is constant.

EXAMPLES:

I. 3, 4, 5: 3, 25, 26: 3, 148, 149: 3, 865, 866: 3, 5044, 5045: &c.
II. 4, 3, 5: 4, 13, 15: 4, 51, 53: 4, 191, 193: 4, 723, 725: &c.
III. 5, 3, 4: 5, 29, 30: 5, 291, 292: 5, 2885, 2886: 5, 28563, 28564: &c.
Also;5, 5, 6: 5, 51, 52: 5, 509, 510: 5, 5043, 5044: 5, 49925, 49926: &c.
And;5, 12, 13: 5, 122, 123: 5, 1212, 1213: 5, 12002, 12003: 5,118112,118113: &c.
IV. 6, 25, 29: 6, 481, 485: 6, 8665, 8669: 6, 155521, 155525: &c.
V. 7, 15, 20: 7, 169, 174: 7, 1695, 1700: 7, 16801, 16806: &c.
VI. 8, 15, 17: 8, 123, 125: 8, 975, 977: 8, 7683, 7685: 8, 60495, 60497: &c.
VII. 9, 10, 17: 9, 73, 80: 9, 442, 449: 9, 2593, 2600: 9, 15130, 15137: &c.
VIII. 10, 17, 21: 10, 35, 39: 10, 273, 277: 10, 4049, 4053: &c.
IX. 11, 25, 30: 11, 267, 272: 11, 2665, 2670: 11, 26403, 26408: &c.
X. 12, 17, 25: 12, 365, 373: 12, 6617, 6625: 12, 118805, 118813: &c.
XI. 13, 20, 21: 13, 518, 519: 13, 13460, 13461: 13, 349454, 349455: &c.
XII. 14, 61, 65: 14, 1125, 1129: 14, 20221, 20225: 14, 362885, 362889: &c.
XIII. 15, 7, 20: 15, 28, 41: 15, 106, 119: 15, 1015, 1028: &c.
XIV. 16, 25, 39: 16, 241, 255: 16, 1945, 1959: 16, 15361, 15375: &c.
XV. 17, 25, 26: 17, 144, 145: 17, 841, 842: 17, 4904,4905: 17,28585,28586: &c.
XVI.
XVII. 19, 20, 37: 19, 153, 170: 19, 932, 949: 19,5473, 5490: 19,31940, 31957: &c.
XVIII. 20, 53, 55: 20, 1067, 1069: 20, 21305, 21307: 20, 425051, 425053: &c.
XIX. 21, 13, 20: 21, 82, 89: 21, 493, 500: 21, 2890, 2897: 21, 16861, 16868: &c.
XX.
XXI. 23, 123, 130: 23, 2768, 2775: 23, 60843, 60850: 23, 1336848, 1336855: &c

XXII. 24, 35, 53: 24, 87, 105: 24, 679, 697: &c. And; 24, 7, 25: 24, 203, 221: 24, 3367, 3385: &c.

The present is a very good opportunity to add a few of the many series wherein 25 can be employed as *base* of *every* triangle in *every* series.

XXIII. *a*—25, 39, 40: 25, 1923, 1924: 25,96135,96136: 25,4804851,4804852: &c.

b—25, 51, 52: 25, 2535, 2536: 25, 126723, 126724: 25, 6333639, 6333640: &c.

c—25, 74, 77: &c. *d*—25, 371, 374: &c.

e—25, 34, 39: 25, 348, 353: 25,3466,3471: 25,34332,34337: 25,339874,339879:&c.

f—25, 12, 17: 25, 106,111: 25,1068,1073: 25,10594,10599: 25,104892,104897: &c.

g—25, 29, 36: 25, 153, 160: 25, 777.8, 784.8: 25,3902.76,3909.76: 25,19527.752, 19534.752: 25, 97652.7504, 97659.7504: &c.

h—25, 17, 26: 25, 1360, 1369: 25, 95489, 95498: &c.

i—25, 136, 145: 25, 9809, 9818: : 25, 686800, 686809: &c.

j—25, 17, 28: 25, 63, 74: : 25, 2030, 2041: 25, 18497, 18508: &c.

k—25, 14, 25: 25, 52, 63: 25, 182, 193: 25, 1697, 1708: 25, 15470, 15481: &c.

l— : 25, 741, 754: : 25, 1532261, 1532274: &c.

m— : 25, 101, 114: 25, 3265$\frac{3}{4}$, 3278$\frac{3}{4}$: 25, 7185$\frac{3}{4}$, 7198$\frac{3}{4}$: &c.

n—25, 25, 40: 25, 149, 164: 25, 773.8, 788.8: 25, 3898.76, 3913.76: 25, 19523.752, 19538.752: 25, 97648.7505, 97663.7504: &c.

o—25, 39, 56: : 25, 340, 357: : 25, 38311, 38328: &c.

p—25, 7, 24: : 25, 84, 101: : 25, 10119, 10136: &c.

q—25, 11, 30: 25, 2379, 2398: 25, 310475, 310494: 25, 40360587, 40360606: &c.

r—25, 33, 52: 25, 5393, 5412: 25, 702273, 702292: 25, 91291313, 91291332: &c.

s—25, 113, 132: 25, 15873, 15892: 25, 2064593, 2064612: 25, 268382433, 268382452: &c.

t—25, 267, 286: 25, 35915, 35934: 25, 4669899, 4669918: 25, 607052171, 607052190: &c.

u— : 25, 1129, 1150: 25, 18607, 18628: &c.

v—25, 17$\frac{1}{3}$, 38$\frac{1}{3}$: 25, 59$\frac{1}{3}$, 80$\frac{1}{3}$: 25, 92, 113: 25, 1658, 1679: &c.

w—25, 25, 48: 25, 339, 362: 25, 3457, 3480: 25, 34323, 34346: 25, 339865, 339888: &c.

x—25, 3, 26: 25, 97, 120: 25, 1059,1082: 25,10585,10608: 25,104883,104906: &c.

y—25, 16, 39: 25, 246,269: 25,2536,2559: 25,25206,25229: 25,249616,249639: &c.

z—25, 6, 29: 25, 136, 159: 25, 1446, 1469: 25, 14416, 14439: 25,142806,142829: &c.

&—25, 51, 74: 25, 601, 624: 25, 6051, 6074: 25, 60001, 60024: 25, 594051, 594074: &c.

APPLICATION OF THE NEW EDUCATION TO THE DIFFERENTIAL AND INTEGRAL CALCULUS.

By FLETCHER DURELL, Ph. D., Professor of Mathematics, Dickinson College, Carlisle, Pennsylvania.

[Read before the New York Mathematical Society, November 4, 1893.]

At a recent meeting* of the Physical Society of Great Britain, a paper was read by Mr. F. W. Sanderson, in which was presented a method of teaching mathematics in preparation for science teaching. In the paper and in the discussion which followed, the methods of the so-called new education were emphatically endorsed. The student in each new advance is to begin with the concrete object, something which he can see and handle and perhaps make, and go on to abstractions only for the sake of realized advantages. Drawing is to precede formal Geometry (Euclid) and plotting of curves is to precede Analytical Geometry. In the course of the discussion it was remarked that the "plotting of curves might be made a means of furnishing the fundamental notions in the Differential and Integral Calculus". As a matter of fact, the writer has been using a method of this sort for some years in the class room.

It being supposed that the pupil has been made thoroughly familiar with methods of drawing graphs of all ordinary algebraic and trigonometric functions, ideas of continuity and discontinuity having been thus formed to some extent by induction, the subject of the Differential and Integral Calculus may then be approached as follows: Take some simple convenient curve as $y=x^2+1$ ....(1) and trace it. Let the student recall the method of finding the slope of the tangent of this curve, given in most American text-books on Analytical Geometry; that is, take two points on the curve, P or (x_1,y_1), and Q or (x_2,y_2) and obtain the slope of the secant through them, $\frac{y_2-y_1}{x_2-x_1}=x_2+x_1$. Let the point Q move along the curve till it coincides† with P, then must the slope of the tangent at $P=2x_1$. But this method of finding the slope of a curve fails, or, at least, becomes very difficult for many curves as, for example, $y=\sin x^2$. Hence, in order to obtain greater power, we modify the method thus: denote P by (x,y), Q by $(x+\Delta x,y+\Delta y)$, then by a precisely similar use of the equation obtained, we get $\frac{\Delta y}{\Delta x}=2x+\Delta x$...(2), and the slope of the tangent at $P=2x$. The difference between the two methods and the advantage of the second method is seen to lie in this: In the first method the co-ordinates of the two points, P and Q, viz.; x_1,x_2,y_1,y_2, are made primary in the notation, and we are left to work out the differences of these which determine the slope of the secant as best we may. But in the second method, these differences, Δx and Δy are made primary in the notation, and appear in the first equations; hence, the whole work can readily be guided with reference to these differences and the determination of their ratio.

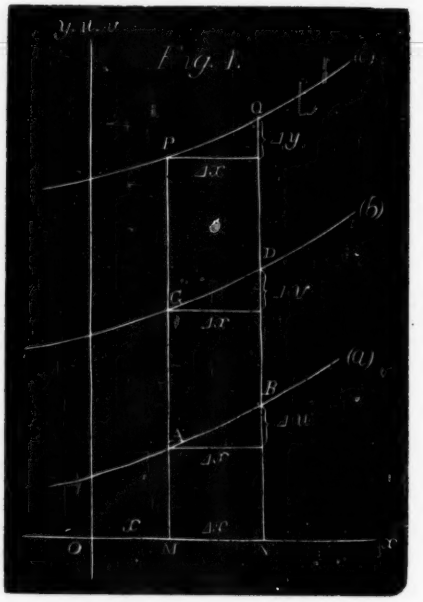
* January 13, 1893. See NATURE, Vol. 47, pp. 358-9.

† See the discussion of the propriety of making the two points actually coincide, given later, pp. 16-17.

In this advantage lies the germ of the Differential Calculus. In order to fully grasp and realize this, let the student find the slopes, by this differential method, of curves like $y=x^3$ and $y=\frac{x+1}{x}$, trace them and draw tangents at indicated points, by use of the slope values obtained.

Differentiation may now be provisionally defined as the method of finding the slope of a curve by the differential method.

We next proceed to apply the differential method to finding the slopes of curves in general groups. In order to do so we suppose general relations to exist among curves thus: in Fig. 1, if each ordinate of (c) equals the sum of the corresponding ordinates of (a) and (b), as $PM=AM+GM, QN=BN+DN$, etc., (c) may be regarded as the sum of (a) and (b) by addition of ordinates. If, instead, the number of units in each ordinate of (c) =the product of the numbers of units in the corresponding ordinates of (a) and (b), (c) may be regarded in like manner as the product of (a) and (b). Thus all the other elementary functional relations as quotient, power, root, etc., may be conceived of as existing among curves.



If now the equation of (a) be $u=f(x)$,

of (b), $\quad v=F(x)$,

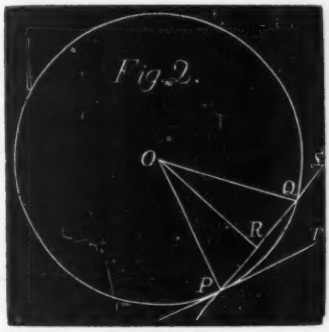
of (c), $\quad y=u+v$,

we obtain as usual, $\frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}$, that is, from Fig. 1, slope sec. PQ=sl. sec. GD+sl. sec. AB. Let Q move to P, then *must* sl. tan. at P=sl. tan. at G+sl. tan. at B, or, using symbols, $\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ....(3). Hence, if the slopes of (a) and (b) are known, that of (c) can be at once written down.

Let us now pause* to notice the matter of letting the secant become a tangent and why when $\Delta y, \Delta x$ and the other differences become zero, we can still determine the slopes of tangents at P, A, and G. Does the secant actually become a tangent or does it only approach the tangent as a limiting position? Some writers say that it actually becomes a tangent; others that it only approaches it as a limit; still others attempt to make compromises, but with results

*This digression is not necessary in teaching a class if the matter has been properly presented in teaching Analytical Geometry.

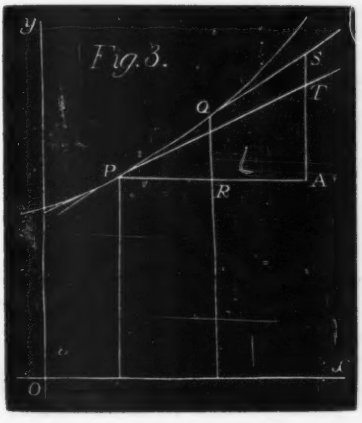
which are puzzling and contradictory.* But it is plain that a secant can become a tangent and that theorems can be directly demonstrated thereby. To give a simple illustration we will prove, by the use of this principle, that the radius of a circle is perpendicular to the tangent at its extremity. Draw the secant PQ of a given circle, OP and OQ radii, and OR the perpendicular from the center O on PQ. Since OPQ is an isosceles triangle, OR bisects the base PQ, $\therefore$ R lies between P and Q. Let Q move along the circumference to P; when Q coincides with P, R also coincides with P; that is, OR the perpendicular on the secant becomes OP, the perpendicular on the tangent, and at the same time becomes the radius to the point of contact; hence, the radius of the point of contact is perpendicular to the tangent.



Similarly, if the secant of a curve through the points (x_1,y_1) and (x_2,y_2) have an intercept on the axis of x equal to $\frac{3}{x_1+x_2}$, the intercept on the same axis of the tangent at (x_1,y_1) is $\frac{3}{2x_1}$.

Thus a property of a tangent may be obtained directly from the property of the secant, by transforming the secant into a tangent.

But in some cases a modification of this method needs to be introduced. In the Calculus, we need to use negative or indirect reasoning also. Greater generality of treatment and greater facility of transformation and reduction are obtained by determining the direction of the secant by the use of the quantities QR and PR (Fig. 3) than in any other way; the slope of the secant is then represented by $\frac{QR}{PR}$ which appears in symbols as $\frac{\Delta y}{\Delta x}$, $\frac{\Delta u}{\Delta x}$, $\frac{\Delta v}{\Delta x}$ etc. These ratios become $\frac{0}{0}$ when Q is made to coincide with P. If $PA=1$, SA represents the slope of the secant PQ, and TA that of the tangent at P. When the secant becomes the tangent at P, SA becomes TA directly; but inasmuch as the relation between SA and TA (and similarly in similar cases) is proved by means of QR and PR which vanish, our method of proving the value of SA must be negative or indirect. The equation which contains PR and QR as finite quantities

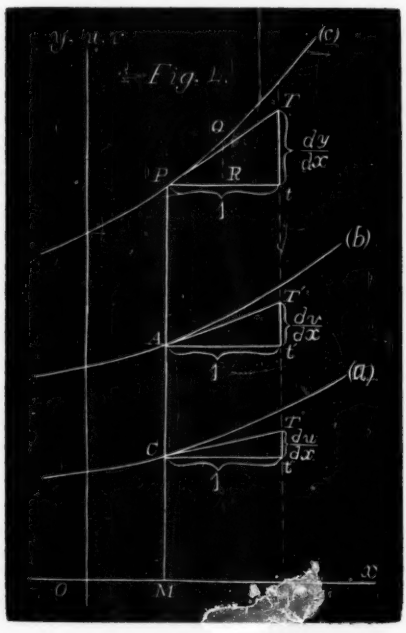


* Thus in the Differential Calculus, by J. Edwards, p. 7, we find the following defini-

shows that the value of TA can be neither greater nor less than what the value of the slope of the secant becomes and therefore equals it. Thus in the case of $y=x^2+1$, given above, the slope of the tangent at (x,y) must $=2x$, for if it $=2x \pm h$, the so-called tangent, by equation (2), would be the secant through the points whose abscissas are x and $x+h$, or x and $x-h$,* and hence would not be the tangent at all. It is in the sense of this negative reasoning that the word "must" is used in the above demonstration. Observe that in the theorem proved from Fig. 2, the auxiliary quantities OP and OQ do not vanish, hence, direct reasoning is used throughout; but in proving a theorem from the data of Fig. 3, the auxiliary quantities PR and QR vanish, and hence negative reasoning is employed.

The advantage of this over the ordinary presentation by the method of limits is that the latter is too often a mere approximation in the mind of the student, while in the method here presented, there is, so to speak, absolute *contact* between the limit and the determining varaible, as well as continuity between other parallel variables and the same limit.

The result obtained in (3) may now be expressed thus, that the student may more firmly grasp the relation of the quantities involved: $Tt=T't'+T''t''$, $\Delta y, \Delta x$, etc., are seen to be merely the auxiliary lines PR, QR, etc., by which the result is established. The student should verify this result by several constructions of curves and their tangents.



tion of a tangent: "Let PQ be a chord joining P and Q, two adjacent points on a curve. Let Q travel along the curve towards P, and come so close as *ultimately to coincide with P*. Then the *limiting position of PQ*, viz.; PT is called the tangent at P."

Salmon says, Conic Sections, Sixth Ed., p. 78, "we shall not say that the tangent meets the circle in only one point, but rather that it meets it in *two coincident* points.... And in general we define the tangent to any curve as the line joining *two indefinitely near points* on the curve."

It will appear in the subsequent investigation that the definition which we provisionally make of a tangent is contained in the first sentence of Mr. Edwards' definition, omitting the word "ultimately." The use of this word, derived though it be from Newton, seems like trying to get rid of a difficulty by postponing it indefinitely in time.

* At a cusp point, this reasoning is somewhat modified. At this point the tangent on being rotated backward must at once become a secant, cutting the curve in an initially adjacent point; hence the slope of the tangent must be such that on being in the slightest degree increased or decreased, in one of these cases it becomes the slope of a secant.

If now, using Fig. 1 in a new way, (c) be taken as the product of (a) and (b) by multiplication of ordinates, we readily obtain, by a similar method of proof, the result,

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}.$$

This result is expressed to the eye by a figure similar to Fig. 4, wherein the number of linear spaces in Tt=(the number of linear spaces in CM)×(the number in $T't'$)+(number in $T''t''$)×(the number in AM).

Thus when the slopes of two curves are known, that of their product can be at once written down.

In this way all the ordinary formulas of differentiation are obtained as applied to curves.

The student should now be introduced to the idea of function, that is of quantity independent of any spacial relations, and brought to see the greater freedom and flexibility of treatment obtained by its use.

(TO BE CONTINUED.)

ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

1. I. Reduce $\frac{1}{3}$ to 8ths.

1st Solution.

II. 1. $1=8$ eighths.

2. $\frac{1}{3}=\frac{1}{3}$ of 8 eighths$=2\frac{2}{3}$ eighths.

III. $\therefore \frac{1}{3}=2\frac{2}{3}$ eighths, or $\frac{2\frac{2}{3}}{8}$.

2nd Solution.

II. 1. $1=\frac{8}{8}$.

2. $\frac{1}{3}=\frac{1}{3}\times\frac{8}{8}=\frac{2\frac{2}{3}}{8}$.

III. $\therefore \frac{1}{3}=\frac{2\frac{2}{3}}{8}$.

2. I. Reduce $\frac{5}{6}$ to a fraction whose numerator is 11.

II. 1. $1=\frac{11}{11}$

2. $\frac{5}{6}=\frac{5}{6}\times\frac{11}{11}=\frac{11}{6}\frac{5}{5}=\frac{11}{9\frac{1}{6}}$.

III. $\therefore \frac{5}{6}=\frac{11}{9\frac{1}{6}}$, a fraction whose numerator is 11.

3. **Proposed by L. B. Hayward, Superintendent of Schools, Bingham, Ohio.**

Bought an article and sold it for 3% less than it cost me; bought it back paying 3% more than I sold it for. I lost $12.00 by the transaction. What did the article cost at first?

Solution by the Proposer.

Let 100% = the cost at first.

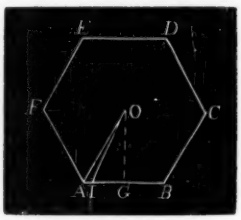
Then 97% = selling price.

And 3% of .97=.02 $\frac{91}{100}$.
97% + $2\frac{91}{100}$% = $99\frac{91}{100}$% = cost on repurchasing.
Loss is 100% − (97 − $2\frac{91}{100}$)% = $5\frac{91}{100}$%.
∴ $5\frac{91}{100}$% = \$12.00.
Then 1% = \$2.0305.
And 100% = \$203.05.

4. I. What is the number of acres in a field in the form of a regular hexagon, if it contains as many acres as there are boards in the fence inclosing it, the boards being $l = 8\sqrt[4]{3}$ feet long and the fence $n = 6$ boards high?

Solution by B. F. FINKEL, Professor of Mathematics in Kidder Institute, Kidder, Missouri.

Construction.— Let $ABCDEF$ be the field, O the center, AI the length of a panel of the fence. Connect A and I with the center of the field by the lines AO and IO. Draw OG perpendicular to the side AB. Then



II.
1. $AI = 8\sqrt[4]{3}$ feet, and
2. *area* of $AOI = 6A = 261360$ sq.ft., since there are as many acres in the field as there are boards in the fence inclosing it and the panel being 6 boards high.
3. $\frac{1}{2}(AI \times OG)$ = the area of the triangle $AOI = \frac{1}{2}(8\sqrt[4]{3} \times OG) = 4\sqrt[4]{3} \times OG$.
4. $\therefore 4\sqrt[4]{3} \times OG = 261360$, the number of square feet in the triangle AOI, whence
5. $OG = 261360 \div 4\sqrt[4]{3} = \frac{65340}{4\sqrt[4]{3}}$. But
6. $OG = \sqrt{[AO^2 (= AB^2) - AG^2 (= \frac{1}{2}AB)^2]} = \frac{1}{2}AB\sqrt{3}$.
7. $\therefore \frac{1}{2}AB\sqrt{3} = \frac{65340}{4\sqrt[4]{3}}$, whence
8. $AB = \frac{130680}{\sqrt[4]{3^3}}$ ft., the length of a side of the field. Then
9. $\frac{130680}{\sqrt[4]{3^3}} \div 8\sqrt[4]{3} = 5445$, the number of panels on a side, and
10. $6\ (6 \times 5445) = 196020$, the number of acres in the field.

III. ∴ There are 196020 acres in the field.

Remark.— If we let l = the length of a rail and n, the number of rails in a panel, the number of acres in a hexagonal field will be $174240n^2 \div l^2\sqrt{3}$. From the nature of the problem, n must be integral. ∴ The number of acres and the length of the rails can not both be rational. The above solution is also applicable when the field is in the form of a square.

PROBLEMS.

5. **Proposed by E. E. KINNEY, Anaconda, Montana.**

A board is 16 in. long and 9 in wide. How may it be cut in two parts that the parts joined together may form a square?

6. **Proposed by B. F. FINKEL, Professor of Mathematics in Kidder Institute, Kidder, Missouri.**

What is the volume of a regular pentagonal pyramid, each side of whose base is 10 feet and the altitude 20 feet?

7. If an article had cost me 10% less, the gain would have been 12% more; what was the gain per cent.? [Selected from *Brook's Higher Arithmetic.*]

8. **Proposed by EARL D. WEST, West Middleburg, Logan County, Ohio.**

The number of men in a side rank of a solid body of militia is to the number of men in the front rank as 2 is to 3; if the length and breadth be increased so as to number each 4 men more, the whole body will then contain 2320 men. How many men in the militia?

9. **Proposed by O. S. KIBLER, Superintendent of Schools, West Middleburg, Logan County, Ohio.**

Four logs of uniform thickness whose diameters are each 4 feet, lie side by side and touch each other. In the crevices of these logs lie three logs 3 feet in diameter, and in the crevices of the three logs lie two logs whose diameters are 2 feet. What must be the diameter of a log to lie on the top of the pile and touch the two logs and the middle one of the three logs?

10. **Proposed by MISS LECTA MILLER, B. L., Professor of Natural Science and Art, Kidder Institute, Kidder, Missouri.**

A carpenter is obliged to cut a board, that is in the form of a trapezoid, crosswise into two equivalent parts. The board is 12 ft. long, 2 ft. wide at one end, and one foot wide at the other. How far from the narrow end must he cut?

11. **Proposed by L. B. HAYWARD, Superintendent of Schools, Bingham, Ohio.**

What length of rope will be required to draw water from a well, it being 38 feet to the water, the sweep to be supported by an upright post 20 feet high, and standing 20 feet from the well, and the foot of the sweep to strike the ground 20 feet from the foot of the upright post?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

1. **Proposed by W. L. HARVEY, Portland, Maine.**

$(2x^2-1)^2-(2x^2-4x-1)(2x^2-1)=1$; find the value of x by quadratics.

Solution by the Proposer.

Performing the opperations indicated and collecting, the equation reduces to $8x^3-4x=1$. Multiplying both sides of this equation by $8x$ and adding $16x^2+1$ to both sides of the resulting equation, we have $64x^4-16x^2+1=16x^2+8x+1$, from which $8x^2-1=4x+1$. Then $8x^2-4x=2$, from which $x=\frac{1}{4}(1\pm\sqrt{5})$. Using the minus sign this will prove true.

2. **Proposed by Professor P. H. PHILBRICK, C. E., Lake Charles, Louisiana.**

Find x from the equation, $x^3+18x=1529$.

Solution by the Proposer.

Multiply by x, then $x^4+18x^2=1529x=139\times 11x$. Again, x^4+139x^2+

$\left[\frac{139}{2}\right]^2 = 121x^2 + 139 \times 11x + \left[\frac{139}{2}\right]^2$. Extract square root and have,

$x^2 + \frac{139}{2} = 11x + \frac{139}{2}$ or $x^2 = 11x$.

PROBLEMS.

3. Proposed by Professor H. A. WOOD, A. M., Hoboken, New Jersey.

If $x^6 - y^6 = 665$, and $x^3y + xy^3 = 78$, find x and y.

4. Proposed by L. E. PRATT, Tecumseh, Nebraska.

If Σm, Σm^3, Σm^5, ... Σm^{2n-1} are the sums of the 1st, 3rd, 5th,..... $(2n-1)$th powers of the first m natural numbers, prove that $n\Sigma m^{2n-1} + \frac{n(n-1)(n-2)}{\lfloor 3} \Sigma m^{2n-3} + \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5} \Sigma m^{2n-5} + ... = 2^{n-1}\Sigma_m^n$.

5. Proposed by WILLIAM E. MAY, Jonesboro, Tennessee.

A, B, and C went to market, each with 10, 30, and 50 eggs, respectively. On their way to market, they agreed to sell their eggs at the same price per dozen so as to realize an equal integral number of cents. How much did they receive?

6. Proposed by L. E. PRATT, Tecumseh, Nebraska.

A vessel is to be filled with water by two pipes. The first pipe is kept open during m-nth of the time which the second would take to fill the vessel; then the first pipe is closed and the second is opened. If the two pipes had kept open together, the vessel would have been filled t hours sooner, and the first pipe would have brought in p-qth of the quantity of water which the second pipe really brought in. How long would it take each pipe alone to fill the vessel?

7. Proposed by O. S. KIBLER, Superintendent of Schools, West Middleburg, Logan county, Ohio.

A's age equals B's age plus the cube root of C's age; B's age equals C's age plus the cube root of A's age plus 14 years; and, C's age equals the cube root of A's age plus the square root of B's age. What is the age of each?

8. Proposed by H. M. CASH, Salesville, Ohio.

The longer side BC of a field in the form of a parallelogram is a (78) rods; the sum of its shorter side AB, and greater diagonal AC is b (114) rods; the distance from B at right angles with AB to a tree standing on AC, is c (32) rods. Find the area of the field, and the distance from the tree to the corners A, C, and D.

GEOMETRY.

Conducted by B. F. FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

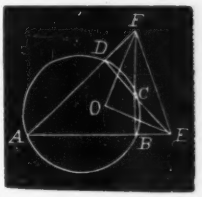
SOLUTIONS TO PROBLEMS.

1. Proposed by B. F. FINKEL, Professor of Mathematics in Kidder Institute, Kidder, Missouri,

Show that the bisectors of the angles formed by producing the sides of an inscribed quadrilateral intersect each other at right angles.

Solution by the Proposer.

Let $ABCD$ be the inscribed quadrilateral and FO and EO the bisectors of the angles F and E, respectively, formed by producing the sides of the quadrilateral. Denote the angle EAF by A; AFB, by F; BFE, by F'; AED, by E; DEF, by E'; FCE, $=DCB$, by C; and FOE, by O. Then $A+C=2$ rt. angles..(1); being opposite angles of an inscribed quadrilateral. Also, in the triangle AFE, $A+F+F'+E+E'=2$ rt. angles..(2); in the triangle FOE, $\frac{1}{2}F+F'+\frac{1}{2}E+E'+O=2$ rt. angles....(3); and, in the triangle FCE, $F'+E'+C=2$ rt. angles....(4).



Multiplying (3) by two and subtracting (4) from the resulting equation, we have $F+F'+E+E'+2O-C=2$ rt. angles..(5). Subtracting (5) from (2), we have $A+C-2O=0$, whence $2O=A+C=2$ rt. angles. $\therefore O=$a rt. angle.

Q. E. D.

PROBLEMS.

2. Show that $\frac{1}{2}\pi=\left[\frac{2.\ 4.\ 6.\ 8.\ 10}{1.\ 3.\ 5.\ 7.\ 9}\cdots\cdots\right]^2$, *Wallis's* expression for π.

[Selected from *Bowser's Trigonometry.*]

3. If A be the area of the circle inscribed in a triangle, A_1, A_2, A_3 the areas of the escribed circles, show that $\frac{1}{\sqrt{A}}=\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}$.

[Selected from *Todhunter's Plane Trigonometry.*]

4. Three circles whose radii are a, b, and c touch each other externally; prove that the tangents at the points of contact meet in a point whose distance from any one of them is $\left\{\frac{abc}{a+b+c}\right\}^{\frac{1}{2}}$. [Selected from *Todhunter's Plane Trigonometry.*]

5. **Proposed by ADOLPH BAILOFF, Durand, Wisconsin.**

If from a varible point in the base of an isosceles triangle, perpendiculars are drawn to the sides, the sum of the perpendicular is constant and equal to the perpendicular let fall from either extremity of the base to the opposite side.

6. **Proposed by EARL D. WEST, West Middleburg, Logan county, Ohio.**

Having given the sides 6, 4, 5, and 3 respectively of a trapezium, inscribed in a circle, to find the diameter of the circle.

7. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in the Ohio University, Athens, Ohio.**

Through each point of the straight line $x=my+h$ is drawn a chord of the parabola $y^2=4ax$, which is bisected in the point. Prove that this chord touches the parabola $(y-2mn)^2=8a(x-h)$.

8. **Proposed by ADOLPH BAILOFF, Durand, Wisconsin.**

If the two exterior angles at the base of a triangle are equal, the triangle is isosceles.

9. Proposed by J. C. GREGG, Brazil, Indiana.

Two circles intersect in A and B. Through A two lines CAE and DAF are drawn, each passing through a centre and terminated by the circumferences. Show that $CA \times AE = DA \times AF$. [*Euclid.*]

10. Proposed by ERIC DOOLITTLE, Instructor in Mathematics, State University of Iowa, Iowa City.

If MN be any plane, and A and B any point without the plane, to find a point P, in the plane, such that $AP+PB$ shall be a minimum.

11. Proposed by Miss LECTA MILLER, B. L., Professor of Natural Science and Art, Kidder Institute, Kidder Missouri.

A gentleman's residence is at the center of his circular farm containing $a = 900$ acres. He gives to each of his $m=7$ children an equal circular farm as large as can be made within the original farm; and he retains as large a circular farm of which his residence is the center, as can be made after the distribution. Required the area of the farms made.

12. Proposed by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

Let OA and OB represent two variable conjugate semi-diameters of the ellipse $\frac{y^2}{a^2}+\frac{y^2}{b^2}=1$. On the chord AB as a side describe an equilateral triangle ABC. Find the locus of C.

13. Proposed by HENRY HEATON, M. S., Atlantic, Iowa.

Through two given points to pass four spherical surfaces tangent to two given spheres.

14. Proposed by HENRY HEATON, M. S., Atlantic City, Iowa.

Through a given point to draw four circles tangent to two given circles.

15. Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.

A man starts from the centre of a circular 10 acre field and walks due north a certain distance, then turns and walks south-west till he comes to the circumference, walking altogether 40 rods. How far did he walk before making the turn?

16. Proposed by H. C. WHITAKER, B. S., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

Three lights of intensities 2, 4 and 5 are placed respectively at points the coordinates of which are (0,3) (4,5) and (9,0). Find a point in the plane of the lights equally illuminated by all of them.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

PROBLEMS.

1. Find the moment of inertia about the origin, of the area included within the parabola $y^2=4ax$, the line $x+y=4a$, and the axis of x.

[Selected from *Osborne's Differential and Integral Calculus.*]

2. Proposed by E. S. LOOMIS, A. M., Ph. D., Professor of Mathematics, Baldwin University, Berea, Ohio.

Show that the indeterminate form $\frac{x-\frac{2}{3}\sin x-\frac{1}{3}\tan x}{x^5}=\frac{-1}{20}$, when $x=0$.

[Ex. 51, p. 112, *Williamson's Differential Calculus.*]

3. Proposed by H. C. WHITAKER, B. S., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

The product of two sides of a triangle is $6000(k^2)$, the length of the bisector of the included angle is 60 (b). What is the maximum area of the triangle, what is the greatest length of the third side, and what the total length swept over by the triangle, the bisector remaining fixed? [Selected from *Philadelphia Call*, 26 May, 1890.]

4. Proposed by J. M. COLAW, Principal of High School, Monterey, Virginia.

Three towns A, B, and C are in the same straight line. The distance from A to B is 20 miles and the distance from B to C is 80 miles. A pedestrian started from B for C and traveled at the variable rate of 10 miles an hour reciprocally as the cube root of his distance from A. In what time did he travel from B to C?

5. Proposed by CHARLES E. MYERS, Canton, Ohio.

The volume generated by the curve whose equation is $y^2=px$, revolving about its axis, is cut by a right cylinder whose equation is $y^2=px-x^2$, the axis of the latter passing through the focus of the former. Find the volume common to both by the formula $V=\int\int\int dxdydz$.

6. Proposed by O. S. KIBLER, Superintendent of Schools, West Middleburg, Ohio.

A string is wound spirally twenty times around a cylinder 20 feet high and 2 feet in diameter. Through what distance will a dove fly in unwinding the string keeping it tense at all times (1) flying in the same plane and (2) not flying in the same plane?

MECHANICS.

Conducted by B. F FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

PROBLEMS.

1. Proposed by A. M. SCRIPTURE, A. M., Principal of Schools, New Hartford, New York.

Suppose A and B to start at the same time from the same point in a level plane. A walks south, going 30 inches at a step, stepping twice in a second and beating a drum at the begining of each step. B steps west 30 inches every time he hears the drum beat. How far apart will they be at the end of 15 minutes, if the temperature of the air is $+41^\circ$ Fahrenheit?

2. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, California.

The axis of a parabola coincides with a vertical line. What is the position of that focal chord through which a body would roll down in the least time?

3. Proposed by CHARLES E. MYERS, Canton, Ohio.

A spherical air-bubble, having risen from a depth of 1,500 feet in water, was one inch in diameter when it reached the surface; what was its diameter at the point of starting?

4. Proposed by DeVOLSON WOOD, M.A.,C. E., Professor of Mechanical Engineering,Stevens Institute of Technology, Hoboken, New Jersey.

A particle starts at rest and revolves in a circle with a uniform acceleration, acquiring a velocity v in t seconds. Find the locus of the foot of the perpendicular from the centre of the circle upon the resultant acceleration.

5. Proposed by J. R. BALDWIN. A. M., Professor of Mathematics and Commercial Law, Davenport Business College, Davenport, Iowa.

A 200 pound ball lies on a three legged table, having the legs equally distant apart and perpendicular to the plane of the top of the table. (1) What is the weight on each leg of the table not including the top when the ball is 2 feet, 3 feet, and 4 feet distant from the three legs? (2) If the ball is 2 feet, 3 feet, and 5 feet from the legs, what must be the weight of the top to keep from tipping and the weight on each leg excluding the top and also including the top?

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

PROBLEMS.

1. Proposed by EARL D. WEST, West Middleburg, Logan County, Ohio.

It is required to divide a given square number into two such parts that each part will be a square number.

2. Proposed by J. M. COLAW, Principal of High School, Monterey, Virginia.

Find two numbers, such that the difference of their squares may be a cube, and the difference of their cubes a square.

3. Proposed by O. S. KIBLER, Superintendent of Schools, West Middleburg, Logan County, Ohio.

It is required to find three whole numbers in an arithmetical progression, such that the sum of every two of them shall be a square.

4. Proposed by H. W. HOLYCROSS, Superintendent of Schools, Pottersburg, Union County, Ohio.

What value of x will render $4x^4+12x^3-3x^2-2x+1$ a square?

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

PROBLEMS.

1. **Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

Three persons A, B, C, throw with three dice. They each stake \$10.00 and the one who first throws at least ten with the three dice takes the whole stake. Find the expectation of each.

2. **Proposed by O. S. KIBLER, Superintendent of Schools, West Middleburg, Logan County, Ohio.**

What is the average area of a triangle formed by joining an angle of a square with any two points within the square?

3. **Proposed by MISS LECTA MILLER, B. L., Professor of Natural Science and Art, Kidder Institute, Kidder, Missouri.**

A deer, wounded at the corner of a square park, is equally liable to run in a straight line in any direction, from the corner of the park, and, at the same time, is also equally liable to drop dead before running a distance equal to the diagonal of the park. What is the chance that the deer will drop dead in the park?

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

PROBLEMS.

1. **Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

To divide the arc of a cycloid into eight equal parts.

2. **Proposed by SYLVESTER ROBINS, Long Branch Depot, New Jersey.**

Give the dimensions of thirteen rational trapezoids each one having 1885 for its parallel bisector; and as many more wherein each bisector is 1105.

3. **Proposed by J. A. CALDERHEAD, Limaville, Ohio.**

Given the simultaneous angular velocities of a body about the principal axes through its center of inertia, find the position of these axes in space at any assigned instant.

4. **Proposed by J K ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.**

I have two circular grindstones, each $\frac{1}{2}$ in. thick. One is 6 in. and the other $4\frac{1}{2}$ in. in diameter, the aperture at center of each being $1\frac{1}{2}$ in. If when in motion they are continually tangent to each other, and $\frac{1}{2}$ cu. in. is ground off the larger wheel and $\frac{1}{4}$ cu. in. off the smaller in the first hour, how must their speed be increased so that the same amount per hour may be ground off each wheel until one is worn out? If in the first hour the larger wheel makes a revolutions, and the smaller b, how many must each make in each succeeding hour?

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

The difinition of the root of an equation is that it must satisfy the equation if substituted in it, that is, that it must produce an identity. But in the equation

$\sqrt{x+4}-\sqrt{x-4}=4$, the value 5, which we get by solving, does not produce an identity. Please explain. L. B.

Answer: A square root, or any even root, calls for the double sign. The value 5 satisfies the equation if the second radical is taken with the negative sign, that is, if it assumes the form $\sqrt{x+4}+\sqrt{x-4}=4$. Under this more general aspect, viz.; that an even root has *two* signs, the value 5 is correct; as soon, however, as you restrict such radicals to one sign, viz.; that originally given, the value 5 is to be rejected, and the solution becomes impossible.

For instance, take $\sqrt{x+a}\pm\sqrt{x-a}=a$, where a is any whole number. Solving we get $x=\frac{a^2+4}{4}$, and substituting we get, $\frac{a+2}{2}\pm\frac{a-2}{2}=a$ or 2, according as we use + or − sign. From which we see that if $a=2$ either sign will give us a value of x that will satisfy the equation.

If $a>2$, the value of $x=a$ when the + sign is used will satisfy the equation, while if − sign is used the value of the algebraic sum equals 2 or a value less than a.

If $a<2$, the value $x=a$ will satisfy the equation for the − sign, while for the + sign the algebraic sum will be greater than a.

In the example, $a=4$, a value greater than 2, and we get $\frac{4+2}{2}-\frac{4-2}{2}=2$.

J. M. C.

"It is claimed that $\frac{1}{203}$ is the probability required in the following problem: There are 30 numbers, 1, 2, 3, 4, etc., in a box from which 6 numbers are drawn at random. What is the probability that the numbers 5, 7, and 12 will be included in the 6 numbers?

I would like to see a full explanation of it, if it is true.

W. L. Harvey, Portland, Maine.

Solution.—The number of combinations that can be formed from 30 numbers, taking 6 at a time, equals the number of ways six numbers can be drawn from 30 numbers. ∴ the number of ways 6 numbers can be drawn from 30 numbers $=\frac{30.\ 29.\ 28.\ 27.\ 26.\ 25}{1.\ \ 2.\ \ 3.\ \ 4.\ \ 5.\ \ 6}=n$

If the numbers 5, 7, and 12 are taken out their still remain 27 numbers. The number of ways that 6 numbers can be drawn from 30 numbers, including 5, 7, and 12, equals the number of ways 3 numbers can be drawn from 27 numbers, and making up the 6 numbers by taking 5, 7, and 12 with each set of 3 numbers drawn.

∴ The number of ways that 6 numbers can be drawn from 30 numbers, including 5, 7, and 12 $=\frac{27.\ 26.\ 25}{1.\ \ 2.\ \ 3.}=n'$. The required probability $=p=\frac{n'}{n}$.

∴ $p=\frac{27.\ 26.\ 25}{1.\ 2.\ \ 3}\times\frac{1.\ \ 2.\ \ 3.\ \ 4.\ \ 5.\ \ 6}{30.\ 29.\ 28.\ 27.\ 26.\ 25}=\frac{4.\ \ 5.\ \ 6}{30.\ 29.\ \ 28}=\frac{1}{203}$, or the odds are 202 to 1 against the event. J. M. C.

[In answer to Wm. E. May, Jonesboro, Tennessee, we shall endeavor to present in the several issues of the Monthly for 1894 matter of the kind that you suggest. You may be interested in Prof. Durell's article in this number. [J. M. C.

NOTES.

W. B. Smith, Ph. D., Professor of Mathematics in Tulane University of Louisiana, New Orleans, is at work on a 2-volume Calculus. He expects to have the first volume ready for the press by June, 1894. J. M. C.

Dr. D. E. Smith, of Michigan State Normal School, wrote under date of Nov. 6, "it may interest you to know that there is serious talk of starting a periodical in England this fall, devoted to Elementary Mathematics. J. M. C.

Dr. J. H. Boyd is now connected with the department of Mathematics in the University of Chicago. J. M. C.

Macmillan & Co. announce "An Elementary Treatise on Theoretical Machanics" by Prof. Alex. Ziwet, of the University of Michigan, and "A Short Course in the Theory of Determinants" by L. G. Weld, M. A., Professor of Mathematics, State University of Iowa. J. M. C.

A new Mathematical Journal, L'intermidiare des Mathematiciens, is announced to appear in January, 1894. The principal object of this new Monthly is to furnish, in answer to inquiries, information on Mathematical subjects, solutions to problems, etc. The address is Gauthier—Villars, 55, quai des Gauds—Augustins, Paris. The price is 6 francs annually. J. M. C.

EDITORIALS.

This Number of the MONTHLY was delayed on account of securing the plate of Professor Seitz's portrait. We hope to have future numbers to appear promptly each month. We will also make some improvement in the appearance of the Journal.

A number of fine problems and solutions were crowded out of this number. These will appear in February Number.

We are thankful for the encouraging words and generous aid received from so many friends from all parts of the country.

We cordially invite all our subscribers to contribute problems, solutions and papers. We will have them published in their turn.

We earnestly solicit all those into whose hands a copy of this number may fall to remit $2.00, the price of subscription, to B. F. Finkel, Kidder, Missouri, and receive THE AMERICAN MATHEMATICAL MONTHLY for one year. No other Mathematical Journal of the scope, size, and quality has ever been offered to the public at so small a price. Send money by Post Office Money Order or by Draft.

The February Number will contain some very fine papers by noted Mathematicians. We name a few: A new solution to the Celebrated Indeterminate equation $x^2 - Ny^2 = \pm 1$, by A. H. Bell; Note on the Centroid of Plane Areas, by Professor G. B. M. Zerr; Remarks on Division, by J. K. Ellwood;

A Bibliography of the History of Geometry, also a list of Mathematical Periodicals, by Professor R. T. Aley; and, A Note on the Theory of Probability, by Professor G. B. M. Zerr.

Contributors can have copies of their solutions furnished them at very low rates by addressing the Publishers.

We will send THE AMERICAN MATHEMATICAL MONTHLY one year to any person sending us the names of four new subscribers accompanied with $8.00.

If you receive a copy of this issue of the MONTHLY, please examine it thoroughly, and if you wish the MONTHLY to be sent to you regularly, remit $2 at once to B. F. Finkel, Kidder, Mo.

BOOKS AND PERIODICALS.

Uniplanar Algebra, being Part I. of a propædeutic to the Higher Mathematical Analysis. By Irwing Stringham, Ph. D., Professor of Mathematics in the University of California. 8vo, cloth, XIV+146 pp. Price, $1.25.

"From the beginning, with rare exceptions, a singular logical incompleteness has characterized our text-books in elementary algebra. By tradition algebra early became a mere technical device for turning out practical results, by careless reasoning inacuracies crept into the explanation of its principles and, through compilers are still perpetuated as current literature. Thus, instead of becoming a classic, like geometry handed down to us from the Greeks, in the form of Euclid's ELEMENTS, algebra has become a collection of processes exemplified and of principles inadequately explained."

The object of this work is the systematic unfolding of algebra into organic forms, and from a careful examination of the book we can testify that the author has accomplished his object.

In the Introduction, the author treats Proportion by the Geometrical Method. Chapter I. is devoted to the treatment of the Laws of Algebraic operations; Chapter II. to Goniometric and Hyperbolic Ratios; Chapter III., to The Algebra of Complex Quantities; Chapter IV., to Cyclometry; Chapter V., to Graphical Transformations; and Chapter VI., to Properties of Polynomials. We highly recommend the work to those who desire to prepare themselves to read, intelligently, the Higher Mathematical Analysis. B. F. F.

A Laboratory Manual. By G. B. M. Zerr, A. M., Principal Stanton Publi Schools. Price 60 cents. Stanton, Va.: Caldwell and Holt.

The principal features of this Laboratory Manual are that the tabulated statements are short and yet contain all that is necessary to work intelligently. The matter is such as the author has used with his classes for several years. The book though small contains a thorough course in synthetic chemistry of the non-metals and metals, a thorough course in blow-pipe analysis, and a thorough course in qualitative analysis. All matter that is not practical has been omitted. Teacher of chemistry in the high schools and preparatory schools of the country will find this excellent little work by Professor Zerr well adapted to their needs. J. M. C.

Table Book and Test Problems in Mathematics. By J. K. Ellwood, A. M., Principal of the Colfax School, Pittsburg, Pa. 16 mo, cloth, 288 pp. Price $1.00, New York: American Book Comyany, 1892.

Part I. of this book contains a large collection of rules, theorems, formulas, tables of logarithms, sines, tangents, etc., indexed.

Part II. contains about 400 test problems, in Arithmetic, Algebra, Geometry, Trigonometry, all properly arranged and classified. A great variety of fine problems.

Part III. contains solutions corresponding to the problems in the second part. Many of the problems are solved in several ways, so that if one solution is not understood another may be examined. While all the solutions are clear and good, some of them are "gems." The title of this book does not announce all the good things it contains. Every live teacher should have this work on his desk, and every ambitious student of Mathematics should seek its aid.

Algebraic Analysis. Solutions and Exercises Illustrating the Fundamental Theorems and the most Important Processes of Pure Algebra. By G. A. Wentworth, A. M., Professor of Mathematics in Phillips Exeter Academy; J. A. McLellan, LL. D., Inspector of Normal Schools, and Conductor of Teachers' Institutes, for Ontario, Canada; and J. C. Glashan, Inspector of Public Schools, Ottawa, Canada. Part I. concluding with Determinants. 8vo, half leather back, x+418 pp. Price, $1.60. Boston: Ginn & Co.

This work is intended to supply the student of Mathematics with a well filled store house of solved examples and unsolved exercises in the application of the fundamental theorems and processes of pure Algebra, and to exhibit to them the highest and most important results of modern algebraic analysis. The book contains the most extensive collection of problems and solutions in algebra that we have yet seen. Fine solutions are given to many of the most difficult problems. The book contains ten chapters. Chapter I. treats of Horner's Division, &c.; Chapter II.,Principles of Symmetry, &c.; Chapter III.. Factoring; Chapter IV., Measures and Multiples, &c.; Chapter V., Linear Equations of One Unknown Quantity; Chapter VI., Simultaneous Linear Equations; Chapter VII., Quadratic Equasions; Chapter VIII., Indices and Surds; Chapter IX., Cubic and Quartic Equations; Chapter X., Determinants.

The authors promise a second volume which will treat of factorial and combinatory analysis; finite differences and derived functions of single variables; expansions, summation, rev rsion, transformation and interpolation of series; the arithmetic,harmonic, and geometric series of integral orders including the thata-functio s, recurring series; binominal, logarithmic, and exponential series; hyperbolic and circular functions; trigonometric series, direct, indirect, and inverse; Legendre's, Bassel's, Lame's, and Heine's series and their associate functions; &c. &c.

The second volume will be hailed with great joy by the mathematicians in every part of the United States and Canada. B. F. F.

A Mathematical Solution Book. By B. F. Finkel, member of the New York Mathematical Society, and Professor of Mathematics in the Kidder Institute. 8vo, cloth, 352 pp. Price, $2.00. Kidder, Mo.: Kibler, Cokely, & Co., 1893.

In this volume the author has given systematic solutions to many of the most difficult problems contained in the leading text-books on Arithmetic, Algebra, Geometry, Trigonometry, and the Calculus. He has also included many problems with model solutions

This book contains the first attempt to present a systematic treatment of solutions to problems—methods in which the work explains itself—and will prove a great boon to the great body of progressive teachers, who value true training above "short methods," and memorized rules. Great benefit is to be derived by carefully studying solutions of the nature given in this book, and incalculable good will be done in the public schools by following the logical order in presenting solutions to classes.

One of the best features in Prof. Finkel's book is the chapter on Mensuration,

which alone will tell strongly in favor of the book for use among Mathematicians. The etymology of all the principal terms is given, and in order and completness of treatment the book is most satisfactory. The formulæ for Conic Sections, Higher Plane Curves, Spirals, Cylindric and Conical Ungulas, etc., are the best possible. Under "Solutions to Miscellaneous Problems" many fine exhibitions of mathematical skill are given.

This book is not only valuable and servicable to teachers and Mathematicians. but can be used to a great advantage in the class room—the problems at the end of each chapter affording ample exercise for supplementary work. A personal examination is needed to appreciate all the valuable features of the Solution-Book. J. M. C.

Miscellaneous Notes and Querries.

A Monthly Journal of History, Folk Lore, Art, Science, Literature, Masonry. Mysticism, Mathematics, Metaphysics, Theosophy, etc. This magazine contains a large number of the odds and ends in all departments of literature "from many a quaint and curious volume of forgotten lore." Commenced July, 1882. Now in its Twelfth Volume. Each volume fully indexed. $1.00 a year in advance. Its motto is "Many people know many things, no one everything." Circulates in all parts of the world. Address S. C. & L. M. GOULD, Manchester, N. H.

Annals of Mathematics, Ormond Stone, Editor, University of Virginia. Bi-Monthly, $2.00.

In the November number are articles on "Intermediate Orbits,"by G. W. Hill: "Proof of a Formula due to Cauchy," by W. H. Echols; and "Some forms of Lagrange's Interpolation Formula," by W. H. Echols. Five exercises are solved and one new one proposed.

The Bulletin of the New York Mathematical Society for December has the following table of contents: "A Doubly-Infinite System of Simple Groups," by Prof. E. Hastings Moor; "Note on Monogenic Functions of a Single Variable," by Prof. Thomas Craig; "Lambert's Non-Euclidean Geometry," by Prof. Geo. B. Halsted; "The Teaching of Mathematics at Gottingen; "Notes"; and "New Publications."

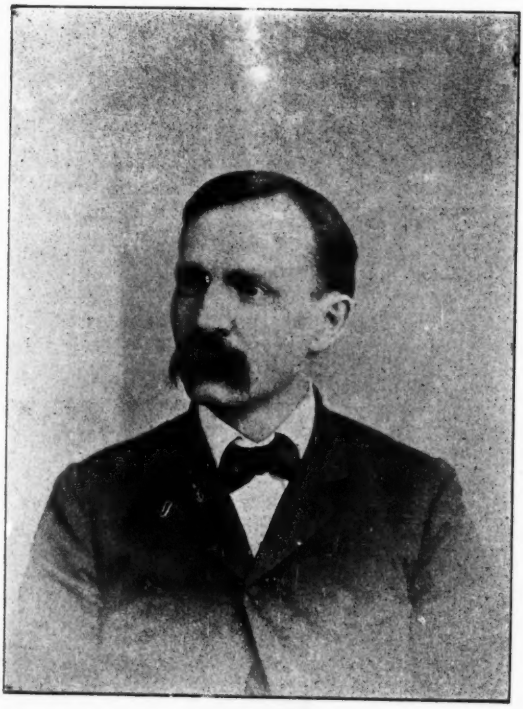
It is to be regretted that neither Dr. Martin's *Visitor*, nor Mr. Harvill's *Messenger* has appeared during 1893.

The Mathematical Magazine, Artemas Martin, LL. D., Editor, Washington, D. C.

Only one number of this valuable periodical (March, 1893) has appeared during 1893. It contains "A table of the square roots of the prime numbers of the form $4m+1$ less 10000, expanded as periodic continued fractions," by C. A. Roberts, with introduction by Dr. Martin, which occupies 16 pages. Five problems are solved and eight new ones proposed. The last pages contain "Editorial Notes," and "Periodicals Received."

The School Visitor, John S. Royer, Editor, Versailles, Ohio.

Friend Royer maintains the interest of the department of "Practical Mathematics." Thirteen interesting problems in elementary Mathematics are solved and seven new problems are proposed in the November Number.



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